

Warped AdS₃ Black Holes

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Abstract

Three dimensional topologically massive gravity (TMG) with a negative cosmological constant $-\ell^{-2}$ and positive Newton constant G admits an AdS₃ vacuum solution for any value of the graviton mass μ . These are all known to be perturbatively unstable except at the recently explored chiral point $\mu\ell = 1$. However we show herein that for every value of $\mu\ell \neq 3$ there are two other (potentially stable) vacuum solutions given by $SL(2, \mathbb{R}) \times U(1)$ -invariant warped AdS₃ geometries, with a timelike or spacelike $U(1)$ isometry. Critical behavior occurs at $\mu\ell = 3$, where the warping transitions from a stretching to a squashing, and there are a pair of warped solutions with a null $U(1)$ isometry. For $\mu\ell > 3$, there are known warped black hole solutions which are asymptotic to warped AdS₃. We show that these black holes are discrete quotients of warped AdS₃ just as BTZ black holes are discrete quotients of ordinary AdS₃. Moreover new solutions of this type, relevant to any theory with warped AdS₃ solutions, are exhibited. Finally we note that the black hole thermodynamics is consistent with the hypothesis that, for $\mu\ell > 3$, the warped AdS₃ ground state of TMG is holographically dual to a 2D boundary CFT with central charges $c_R = \frac{15(\mu\ell)^2 + 81}{G\mu((\mu\ell)^2 + 27)}$ and $c_L = \frac{12\mu\ell^2}{G((\mu\ell)^2 + 27)}$.

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1 Introduction and summary

Topologically massive gravity [1, 2] is described by the action

$$I_{TMG} = \frac{1}{16\pi G} \left[\int d^3x \sqrt{-g} (R + 2/\ell^2) + \frac{1}{\mu} I_{CS} \right] \quad (1.1)$$

where I_{CS} is the gravitational Chern-Simons action (given explicitly below) and we take both G and μ positive¹. For every value of the coupling μ TMG has a classical AdS_3 solution with radius ℓ . For large ℓ (and positive G) the linearized excitations about AdS_3 describe a propagating graviton with positive mass μ , but negative energy (as well as the usual massless positive-energy gravitons). Hence the AdS_3 vacua are generically expected to be unstable and the quantum theory appears ill-defined. However at the critical value $\mu\ell = 1$ the Compton wavelength of the massive graviton reaches the AdS_3 radius and the linearized energy spectrum of asymptotically AdS_3 excitations is non-negative.² Accordingly it was conjectured [10] that a consistent quantum theory of so-called chiral gravity can be defined at $\mu\ell = 1$.

In this paper we shall focus on non-chiral values of $\mu\ell$. The fact that for $\mu\ell \neq 1$ the AdS_3 vacua are all unstable does not preclude the possibility that these theories have other stable ground states around which they can be consistently expanded. With this in mind we look for other vacua and find that there are in fact two warped AdS_3 vacua (some of which were known already in [11, 12]) for every value of μ . Ordinary AdS_3 can be viewed as a fibration of the real line over AdS_2 . Warped AdS_3 is similar but with a constant warp factor multiplying the fiber metric. This breaks the $SL(2, \mathbb{R})_L \times SL(2, \mathbb{R})_R$ isometry group of AdS_3 down to $SL(2, \mathbb{R}) \times U(1)$. The fiber is stretched (squashed) for $\mu\ell > 3$ ($\mu\ell < 3$) and there are solutions with both timelike and spacelike $U(1)$ isometries. At the critical value $\mu\ell = 3$ there are still two warped solutions, but both have a null $U(1)$ isometry.

¹For a recent discussion of the negative G case see [3, 4].

²By asymptotically AdS_3 we mean in the usual sense of Brown and Henneaux [5] or Fefferman and Graham [6]. It was verified in [7] that these boundary conditions are consistent for TMG in that the generators of the asymptotic symmetry group are well-defined finite expressions. It is however possible [8] that there is more than one set of consistent AdS_3 boundary conditions defining inequivalent theories. Recently an alternate definition of the theory with logarithmically weaker boundary conditions, referred to as ‘‘cosmological topologically massive gravity at the chiral point’’, has been proposed [9], but the asymptotic symmetry group or the finiteness of its generators have yet to be studied. It was established [9] that if the theory can be consistently defined in this manner it is unstable, does not have a hermitian Hamiltonian, and is potentially dual to a logarithmic CFT.

The curvature of the warped solutions is of order μ^2 . Hence the Compton wavelength of the massive graviton is always of order the radius of curvature and it cannot be decoupled from that of the background by taking $\mu \rightarrow \infty$ as in the ordinary AdS_3 case. Therefore it is not a priori obvious whether or not the massive gravitons lead to instabilities of the warped vacua. The first (as yet untaken) steps are to understand the boundary conditions for warped AdS_3 , and to solve for the linearized spectrum. The reduced isometry group makes these tasks substantially more difficult than for ordinary AdS_3 . At present we do not know whether or when the warped vacua are perturbatively stable. This is a key issue and we hope to return to it at a later point.

For $\mu\ell > 3$, there are known regular black hole solutions [13, 14, 15] which are asymptotic to warped AdS_3 with a spacelike $U(1)$. We show herein that these warped black holes are discrete quotients by an element of $SL(2, \mathbb{R}) \times U(1)$ of warped AdS_3 , just as BTZ black holes are discrete quotients of AdS_3 [16, 17, 18]. We further describe a fascinating zoo of other solutions of this type for other values of μ .

It is important to note that warped AdS_3 arises in a number of contexts besides TMG, see e.g. [14, 19, 20, 21, 22, 23, 24, 25, 26]. It follows from our quotient construction that the warped black holes are solutions of all these theories as well. In addition warped AdS_3 is a submanifold (at fixed polar angle) of the near horizon geometry of extremal Kerr [27, 23]. We will report elsewhere on the application of our results to Kerr [28].

In AdS_3 the group of elements defining the quotient selects a left and right temperature T_L and T_R of the boundary CFT. Using $c_{L,R} = 3\ell/2G$ (and assuming unitarity), the density of states of the boundary CFT exactly matches the Bekenstein-Hawking entropy of the corresponding black hole [29], thereby explaining the latter. A similar exact match is found for warped black holes in warped AdS_3 provided the central charges are

$$c_R = \frac{15(\mu\ell)^2 + 81}{G\mu((\mu\ell)^2 + 27)} \quad (1.2)$$

$$c_L = \frac{12\mu\ell^2}{G((\mu\ell)^2 + 27)}. \quad (1.3)$$

As this picture fits together rather nicely we cannot resist conjecturing that $\mu\ell > 3$ TMG defined with suitable asymptotically warped boundary conditions exists and is dual to a 2D boundary CFT with central charges 1.2 and 1.3.³ Some non-trivial - but far from definitive - evidence for the conjecture is given herein.

In the next section we define the theory and our conventions. In section 3 we find

³We do not have a conjecture for $\mu\ell \leq 3$ except of course for $\mu\ell = 1$.

the classical warped AdS₃ vacua for all μ . In section 4 we review the known warped black hole solutions. In section 5 we show that they are quotients of warped AdS₃. Section 6 describes the general quotient solution. In section 7 we describe the black hole thermodynamics and formulate a conjecture on the existence of a boundary CFT dual to warped AdS₃.

After this work was completed several works appeared [30, 31] which overlap with section 3 as well as [32, 33, 34] which finds the null black holes of section 6.3 in the context of cold atoms.

2 Framework

Our story begins with the action for three dimensional topologically massive gravity with a negative cosmological constant,

$$I_{TMG} = \frac{1}{16\pi G} \int_{\mathcal{M}} d^3x \sqrt{-g} (R + 2/\ell^2) + \frac{\ell}{96\pi G\nu} \int_{\mathcal{M}} d^3x \sqrt{-g} \varepsilon^{\lambda\mu\nu} \Gamma_{\lambda\sigma}^r \left(\partial_\mu \Gamma_{r\nu}^\sigma + \frac{2}{3} \Gamma_{\mu\tau}^\sigma \Gamma_{\nu r}^\tau \right) \quad (2.1)$$

where $\varepsilon^{\tau\sigma u} = +1/\sqrt{-g}$ is the Levi-Civita tensor and G has the conventional positive sign. The coefficient of the Chern-Simons action involves the dimensionless coupling ν which is related to the graviton mass μ by

$$\nu = \frac{\mu\ell}{3} \quad (2.2)$$

For asymptotically AdS₃ spacetimes there is a critical chiral gravity theory at $\mu\ell = 1$ or $\nu = \frac{1}{3}$ [10]. However our main interest in this paper are the warped AdS₃ vacua. These exhibit critical behavior at $\nu = 1$ or $\mu\ell = 3$. At the risk of confusion we use ν rather than μ as the formulae are significantly simpler. Since the Chern-Simons is parity odd without loss of generality we may take ν to be positive.

Upon varying the above action with respect to the metric we obtain the bulk equation of motion,

$$G_{\mu\nu} - \frac{1}{\ell^2} g_{\mu\nu} + \frac{\ell}{3\nu} C_{\mu\nu} = 0 \quad (2.3)$$

where $G_{\mu\nu}$ is the Einstein tensor and $C_{\mu\nu}$ is the Cotton tensor:

$$C_{\mu\nu} = \varepsilon_\mu^{\alpha\beta} \nabla_\alpha \left(R_{\beta\nu} - \frac{1}{4} g_{\beta\nu} R \right) \quad (2.4)$$

From the equations of motion we see that any Einstein vacuum solution is also a solution of TMG since $G_{\mu\nu} = g_{\mu\nu}/\ell^2$. There are also non-Einstein solutions that solve these equations of motion which we now proceed to explore.

3 $SL(2, \mathbb{R}) \times U(1)$ -invariant TMG vacua

The simplest non-Einstein solution to TMG will be referred to as *warped AdS₃*, as it involves a warped fibration. For every value of ν there are two such solutions, and the structure changes qualitatively at $\nu = 1$. The timelike warped case was previously discovered as a solution of TMG by Nutku and Gürses [13, 14]. It has also been studied in various other contexts [19, 20, 21, 35, 22, 36, 23, 24, 37]. To find the general solution we begin by recalling that AdS₃ can be written as a kind of Hopf fibration over Lorentzian (or Euclidean) AdS₂ where the fiber is the real line [20, 38, 23]. The metric can be written as a spacelike (or timelike) fibration with fiber coordinate u (or τ)

$$ds^2 = \frac{\ell^2}{4} [-\cosh^2 \sigma d\tau^2 + d\sigma^2 + (du + \sinh \sigma d\tau)^2] \quad (3.1)$$

$$= \frac{\ell^2}{4} [\cosh^2 \sigma du^2 + d\sigma^2 - (d\tau + \sinh \sigma du)^2] \quad (3.2)$$

with $\{u, \tau, \sigma\} \in [-\infty, \infty]$ and we have replaced u with $-u$ in 3.2. The $SL(2, \mathbb{R})_L \times SL(2, \mathbb{R})_R$ isometries are given in appendix A. In order to obtain warped AdS₃ we must multiply the fiber by a warp factor, which breaks the isometry group to $SL(2, \mathbb{R}) \times U(1)$, where the $U(1)$ is noncompact. Solutions of this type fall into six different classes.

3.1 Spacelike

A *spacelike* (or hyperbolic) warped anti-de Sitter solution is given by warping 3.1 in the form

$$ds^2 = \frac{\ell^2}{(\nu^2 + 3)} \left[-\cosh^2 \sigma d\tau^2 + d\sigma^2 + \frac{4\nu^2}{\nu^2 + 3} (du + \sinh \sigma d\tau)^2 \right] \quad (3.3)$$

For $\nu^2 > 1$, the warp factor is greater than unity, so we have spacelike *stretched* AdS₃. If $\nu^2 < 1$ we have spacelike *squashed* AdS₃. The isometries preserved by 3.3 are given by $U(1)_L \times SL(2, \mathbb{R})_R$. Note that the solution is not warped at $\nu = 1$.

3.2 Timelike

A *timelike* (or elliptic) warped anti-de Sitter solution is given by a warping of 3.2,

$$ds^2 = \frac{\ell^2}{(\nu^2 + 3)} \left[\cosh^2 \sigma du^2 + d\sigma^2 - \frac{4\nu^2}{\nu^2 + 3} (d\tau + \sinh \sigma du)^2 \right] \quad (3.4)$$

Once again if $\nu^2 > 1$ we have timelike stretched AdS₃. If $\nu^2 < 1$ we have timelike squashed AdS₃. As explored in [19, 23] timelike stretched AdS₃ has closed timelike curves and is essentially the Gödel spacetime. While such spacetimes are quite interesting in their own right, we are limiting the scope of this paper to spacetimes without naked CTCs.

We may also write the squashed metric in the global coordinates⁴

$$\frac{ds^2}{\ell^2} = -dt^2 + \frac{dr^2}{r((\nu^2 + 3)r + 4)} + 2\nu r dt d\theta + \frac{r}{4} (3(1 - \nu^2)r + 4) d\theta^2 \quad (3.5)$$

The origin is at $r = 0$ where the (t, θ) metric determinant vanishes. Regularity requires the identification

$$\theta \sim \theta + 2\pi. \quad (3.6)$$

The coordinate transformation relating the two metrics is a special case of equations 5.3-5.5 below.

3.3 Null

Null (or parabolic) warped AdS₃s [36] are given by the metrics,

$$ds^2 = \ell^2 \left[\frac{du^2}{u^2} + \frac{dx^+ dx^-}{u^2} \pm \left(\frac{dx^-}{u^2} \right)^2 \right] \quad (3.7)$$

with coordinate range $x^\pm \in [-\infty, \infty]$ and $u \in [0, \infty]$. The above metrics are a solution to TMG only for $\nu^2 = 1$, and can be obtained as a kind of pp-wave limit in which $\nu \rightarrow 1$ of the $\nu \neq 1$ warped vacua. Pure AdS₃ is retrieved by dropping the last $(dx^-)^2$ term. The isometry group is $SL(2, \mathbb{R}) \times U(1)_{null}$ where the $U(1)_{null}$ is in the null direction. Null warped AdS₃ has recently appeared in a completely different context in the search for a dual theory for cold atoms [25, 26]. It would be interesting to see whether the rest of the warped anti-de Sitter metrics, or the null black hole solutions

⁴These coordinates are OK locally but have global singularities for the stretched case, as do the spacelike warpings obtained by the double Wick rotation $t \rightarrow it$, $\theta \rightarrow i\theta$

discussed below, are related to this story.

In summary, for $\nu^2 < 1$ there are timelike and spacelike squashed vacua, while for $\nu^2 > 1$ there are timelike and spacelike stretched vacua. At $\nu = 1$ there are two null warped vacua. All of these vacua have an $SL(2, \mathbb{R}) \times U(1)$ isometry. In addition for every ν there is the usual $SL(2, \mathbb{R})_L \times SL(2, \mathbb{R})_R$ -invariant AdS_3 vacuum.

4 Spacelike stretched black holes

We now switch gears and study the black hole solutions which are asymptotic to warped AdS_3 . Solutions which are free of naked CTCs or other pathologies are until now known only for the asymptotically spacelike stretched (i.e. $\nu^2 > 1$) case [15].⁵ These will be reviewed in this section. Closely related black objects were first discussed in [13, 14] and further studied in [39, 15] where their conserved ADM charges [40, 41, 42, 43] and some of their thermodynamic properties were computed.

The metric describing the spacelike stretched black holes for $\nu^2 > 1$ is given in Schwarzschild coordinates by

$$\frac{ds^2}{\ell^2} = dt^2 + \frac{dr^2}{(\nu^2 + 3)(r - r_+)(r - r_-)} + \left(2\nu r - \sqrt{r_+ r_- (\nu^2 + 3)}\right) dt d\theta + \frac{r}{4} \left(3(\nu^2 - 1)r + (\nu^2 + 3)(r_+ + r_-) - 4\nu\sqrt{r_+ r_- (\nu^2 + 3)}\right) d\theta^2 \quad (4.1)$$

where $r \in [0, \infty]$, $t \in [-\infty, \infty]$ and $\theta \sim \theta + 2\pi$. In the ADM form the above metric becomes,

$$ds^2 = -N(r)^2 dt^2 + \ell^2 R(r)^2 (d\theta + N^\theta(r) dt)^2 + \frac{\ell^4 dr^2}{4R(r)^2 N(r)^2} \quad (4.2)$$

where we have defined,

$$R(r)^2 \equiv \frac{r}{4} \left(3(\nu^2 - 1)r + (\nu^2 + 3)(r_+ + r_-) - 4\nu\sqrt{r_+ r_- (\nu^2 + 3)}\right) \quad (4.3)$$

$$N(r)^2 \equiv \frac{\ell^2(\nu^2 + 3)(r - r_+)(r - r_-)}{4R(r)^2} \quad (4.4)$$

$$N^\theta(r) \equiv \frac{2\nu r - \sqrt{r_+ r_- (\nu^2 + 3)}}{2R(r)^2} \quad (4.5)$$

The horizons are located at r_+ and r_- where $1/g_{rr}$ as well as the determinant of the

⁵These will be generalized below for some other asymptotics.

(t, θ) metric vanishes. The vacuum solution for the black holes is given by $r_+ = r_- = 0$ and, like $M = J = 0$ BTZ, is singular at the origin $r = 0$. We also note that the above metric reduces to the metric of the BTZ black hole in a rotating frame when $\nu^2 = 1$. In the parameter region given by $\nu^2 > 1$ we have physical black holes so long as r_+ and r_- are non-negative. For $\nu^2 < 1$ we always encounter closed timelike curves at large values of r when θ is identified. Such geometries will not be considered in this paper.

Finally we mention that the spacelike stretched black hole was studied in a slightly different coordinate system in [15] and we give the details of this metric in appendix C. At this point we have finished describing the basic geometry of the previously known non-Einstein black hole solutions of TMG.

5 Warped black holes as quotients

In this section we will show that the known warped black holes are quotients of warped AdS_3 under a discrete subgroup Γ of the isometry group, much as BTZ black holes are quotients of AdS_3 [16, 17].

5.1 Curvature invariants

A first hint that warped black holes are locally equivalent to warped AdS_3 comes from looking at the coordinate invariant quantities. In three dimensions these are built from the Ricci tensor and its derivatives. It turns out that warped AdS_3 and the warped black hole solutions have the same values for these curvature invariants. The invariants built out of the Ricci tensor are given by,

$$\{R, R_{\mu\nu}R^{\mu\nu}, R_{\mu\nu}R^{\mu\beta}R^\nu{}_\beta\} = \frac{6}{\ell^2} \left\{ -1, \frac{3 - 2\nu^2 + \nu^4}{\ell^2}, \frac{-9 + 9\nu^2 - 3\nu^4 - \nu^6}{\ell^4} \right\} \quad (5.1)$$

The derivatives of the Ricci tensor give rise to the following invariant,

$$\nabla_\mu R^{\beta\alpha} \nabla_\alpha R^\mu{}_\beta = \frac{18\nu^2(\nu^2 - 1)^2}{\ell^6} \quad (5.2)$$

The above agreement between the coordinate invariant quantities of the warped AdS_3 and the warped black holes suggests that they are in fact locally equivalent (see [44] for a rigorous discussion of local equivalence in three dimensions) as we will now demonstrate directly.

5.2 Coordinate transformations

In this subsection we exhibit a local coordinate transformation from warped AdS₃ to the warped black hole metric and thus establish the claim that known warped black holes are locally equivalent to warped AdS₃. When we are in the parameter range $\nu^2 > 1$ the transformation between 3.3 and 4.1 is,

$$\tau = \tan^{-1} \left[\frac{2\sqrt{(r - r_+)(r - r_-)}}{2r - r_+ - r_-} \sinh \left(\frac{1}{4}(r_+ - r_-)(\nu^2 + 3)\theta \right) \right] \quad (5.3)$$

$$\begin{aligned} u = & \frac{\nu^2 + 3}{4\nu} \left[2t + \left(\nu(r_+ + r_-) - \sqrt{r_+ r_-(\nu^2 + 3)} \right) \theta \right] \\ & - \tanh^{-1} \left[\frac{r_+ + r_- - 2r}{r_+ - r_-} \coth \left(\frac{1}{4}(r_+ - r_-)(\nu^2 + 3)\theta \right) \right] \end{aligned} \quad (5.4)$$

$$\sigma = \sinh^{-1} \left[\frac{2\sqrt{(r - r_+)(r - r_-)}}{r_+ - r_-} \cosh \left(\frac{1}{4}(r_+ - r_-)(\nu^2 + 3)\theta \right) \right] \quad (5.5)$$

A similar transformation takes us from 3.4 to 4.1 when we are in the parameter range $\nu^2 < 1$ although the solutions are not regular in that region. The above transformation breaks down for extremal black holes and we present instead the coordinate transformation in appendix B.

5.3 Warped AdS₃ quotients

Since the warped black holes are locally warped AdS₃, they must be a quotient of the latter by a discrete subgroup Γ of the $SL(2, \mathbb{R}) \times U(1)$ isometries - in direct analogy with the case of the BTZ black hole [17]. In this section we find Γ . Let us identify points \mathcal{P} in warped AdS₃ under the action of a Killing vector ξ defining a one parameter subgroup of the full isometry group as follows [17],

$$\mathcal{P} \sim e^{2\pi k\xi} \mathcal{P}, \quad k = 0, 1, 2 \dots \quad (5.6)$$

The isometries of the various types of warped AdS₃ are given in appendix A. From 5.4 we see that in order for the coordinate transformation to reproduce the black hole we must identify points along the ∂_θ direction such that $\theta \sim \theta + 2\pi$. Expressing the ∂_θ Killing vector in terms of the original warped anti-de Sitter coordinates, we

discretely quotient along the isometry

$$2\pi\xi = \partial_\theta = \frac{\nu^2 + 3}{8} \left[\left(r_+ + r_- - \frac{\sqrt{(\nu^2 + 3)r_+ r_-}}{\nu} \right) J_2 - (r_+ - r_-) \tilde{J}_2 \right] \quad (5.7)$$

where the Killing vectors $J_2 \in U(1)_L$ and $\tilde{J}_2 \in SL(2, \mathbb{R})_R$ are given in the appendix. Defining the left and right moving temperatures

$$T_R \equiv \frac{(\nu^2 + 3)(r_+ - r_-)}{8\pi\ell} \quad (5.8)$$

$$T_L \equiv \frac{(\nu^2 + 3)}{8\pi\ell} \left(r_+ + r_- - \frac{\sqrt{(\nu^2 + 3)r_+ r_-}}{\nu} \right) \quad (5.9)$$

we have

$$\partial_\theta = \pi\ell(T_L J_2 - T_R \tilde{J}_2) \quad (5.10)$$

T_L and T_R are referred to as temperatures in analogy with the BTZ case where it is known [45] that the coefficients of the shifts are the temperatures of the dual 2d CFT. In the coordinates 3.3 the norm is

$$\begin{aligned} |T_L J_2 - T_R \tilde{J}_2|^2 &= \frac{12\ell^2(\nu^2 - 1)}{(\nu^2 + 3)^2} [T_R \cos(\tau) \cosh(\sigma)]^2 \\ &+ \frac{32\nu^2\ell^2}{(\nu^2 + 3)^2} T_L T_R \cos(\tau) \cosh(\sigma) \\ &+ \frac{4\ell^2}{\nu^2 + 3} T_R^2 + \frac{16\nu^2\ell^2}{(\nu^2 + 3)^2} T_L^2 \end{aligned} \quad (5.11)$$

Note that for the squashed case the norm is negative at the boundary so a θ quotient would produce CTCs.

In summary the spacelike stretched black hole solution 4.1 is a quotient of spacelike stretched AdS_3 3.3 by a 2π shift generated by the isometry 5.7.

6 More general quotients

In this section we consider more general quotients of warped AdS_3 by discrete subgroups of the isometry group which preserves at least $U(1) \times U(1)$. In appendix A the classification of such discrete subgroups is extracted from [17]. There are a rich variety of possible structures and they differ for the six possible warpings as discussed below.

Of particular interest are quotients that correspond to regular black holes. By this we mean a geometry with a geometric or causal singularity hidden by an event horizon. Geometric singularities arise when the discrete subgroup has a fixed point. Causal singularities arise when the identification produces null or timelike closed curves. In either case the norm of the quotient-generating isometry has a zero. This singularity should be shielded from infinity by an event horizon, i.e. a place where the N^2 multiplying dt^2 in the ADM form of the metric vanishes.

We note that this might not be the only type of geometry which should be thought of as a black hole. For example the boundary of the Poincare patch of AdS_2 or AdS_3 is an event horizon when these spaces arise as near-horizon limits of extremal black holes or strings [46]. There is a Killing horizon associated to time translations but no singularity behind it. These spaces, though free of singularities, can be thought of as black holes. Also one may consider black hole geometries - such as the black holes in a Gödel universe - which have a regular region outside the horizon but CTCs at infinity. Hence the following discussion is a beginning and does not exhaust all interesting possibilities for black holes.

6.1 Spacelike Warped

We begin with the spacelike warped case. The three types of one-parameter subgroups are generated by

$$\eta_a : \quad \beta_2 J_2 + \alpha_0 \tilde{J}_0 \tag{6.1}$$

$$\eta_b : \quad \beta_2 J_2 + \alpha_2 \tilde{J}_2 \tag{6.2}$$

$$\eta_c : \quad \beta_2 J_2 + \tilde{J}_0 + \tilde{J}_2 \tag{6.3}$$

The norms of the Killing vectors generating the subgroups η_a , η_b and η_c are given in A.15, A.16, A.17. The η_c type identifications can be thought of as extremal limits of the previous two cases.

6.1.1 Stretched

We begin with the *stretched* case. If we quotient along η_b with $\alpha_2 \neq 0$ we obtain the warped black hole solution 4.1, as we already discussed. In order to locate the horizon, we go to the coordinate system where $\partial_\phi = \eta_b$, $\partial_t = J_2$ and $r = -\frac{\alpha_2}{\beta_2} \cosh \sigma \cos \tau$.

Then, N^2 in the ADM decomposition is given by

$$N^2 = \frac{(J_2 \cdot \eta_b)^2}{|\eta_b|^2} - |J_2|^2 \quad (6.4)$$

The zeroes of $(N^2|\eta_b|^2)$ give the locations of the horizons, and the zeroes of $|\eta_b|^2$ determine the location of the causal singularities. Explicitly these are located at

$$r_{\pm} \equiv \pm \left| \frac{\alpha_2}{\beta_2} \right|, \quad \text{and}, \quad r_{s\pm} \equiv \frac{-4\nu^2 \pm \sqrt{(\nu^2 + 3)(4\nu^2 - 3(\nu^2 - 1)\frac{\alpha_2^2}{\beta_2^2})}}{3(\nu^2 - 1)} \quad (6.5)$$

where r_{\pm} denote the horizons and r_{s+} denotes the location of the causal singularity. The causal singularity exists only when $\beta_2^2 > 3(\nu^2 - 1)\alpha_2^2/4\nu^2$ and is hidden by the inner horizon for all α_2 and β_2 except when $|\alpha_2| = |\beta_2|$ for which r_- and r_{s+} coincide. When $\beta_2^2 \leq 3(\nu^2 - 1)\alpha_2^2/4\nu^2$ one finds no CTCs upon identification along η_b and there are no singular regions of the spacetime.

Identifying along η_c with $\beta_2 > 0$ gives us the extremal black hole solutions of 4.1 whereas with $\beta_2 < 0$ there are CTCs close to the boundary. Identifying along η_a with $\alpha_0 \neq 0$ gives us naked causal singularities. Finally, identifying along η_a with $\alpha_0 = 0$ gives us no CTCs, and we obtain the self-dual type solutions which we will now describe.

Self-Dual Solutions

When we identify along J_2 we find no CTCs and one could call the quotients *self-dual* solutions in analogy to those quotients of AdS_3 studied in [18]. In Schwarzschild coordinates this corresponds to identifying t , i.e. $t \sim t + 2\pi\alpha$. Thus we can define a left moving temperature $T_L = (\nu^2 + 3)\alpha/(4\pi\nu\ell)$ in the same spirit as 5.9. The right moving temperature vanishes. The entropy of the Killing horizon can be computed using the techniques in [47, 48, 49] and we get:⁶

$$S = S_E + S_{CS} = \frac{\pi\alpha\ell}{3G} = \frac{\pi^2 T_L \ell}{3} \frac{4\nu\ell}{(\nu^2 + 3)G} \equiv \frac{\pi^2 \ell}{3} c_L T_L \quad (6.6)$$

Notice that in Schwarzschild coordinates 4.1, the self-dual solution has a Killing

⁶In section 7 we will define the quantity $c_L \equiv 4\nu\ell/(\nu^2 + 3)G$ in 6.6 as a left moving central charge (together with a similar formula for c_R) and see that the stretched black hole entropy of 4.1 obeys the Cardy formula. Using the same c_L we see that the self-dual solution has entropy obeying the Cardy formula as well.

horizon at the value of r where g^{rr} vanishes. Thus, even though these objects don't fall strictly under the category of regular black holes as defined above, they are in many ways like black holes including their thermodynamic behavior. A similar situation holds for all the self-dual black holes we uncover in what follows. These ones, along with their $\nu < 1$ squashed counterparts, are however of particular interest as they appear as the near horizon region of extremal Kerr at fixed polar angle [27, 23].

6.1.2 Squashed

In the *squashed* case with $\nu < 1$ the norms of the Killing vectors are negative at the boundary unless we identify along η_a with $\alpha_0 = 0$ or along η_b with $\alpha_2 = 0$. Then the quotients become the self-dual type solutions we mentioned.

The entropy of the squashed self-dual solutions is again given by 6.6. Notice that since we are identifying the ∂_u direction, which is always spacelike for spacelike warped AdS, there is no pathology associated with the self-dual solution, even for $\nu < 1$. As we saw in 6.6, such self-dual solutions have an entropy obeying the Cardy formula. Furthermore, they have a Killing horizon in the Schwarzschild coordinates. All these facts suggest that they should be regarded as black holes.

6.2 Timelike Warped

The one-parameter subgroups are given by

$$\xi_a : \quad \alpha_0 \tilde{J}_0 + \beta_2 J_2 \tag{6.7}$$

$$\xi_b : \quad \alpha_0 \tilde{J}_0 + \beta_0 J_0 \tag{6.8}$$

$$\xi_c : \quad \alpha_0 \tilde{J}_0 + J_0 + J_2 \tag{6.9}$$

The norms of the Killing vectors generating the subgroups are given in A.22, A.23 and A.24.

6.2.1 Stretched

In the *stretched* case we already encountered CTCs at infinity for the vacuum solution without any quotienting. Hence there are no regular black holes.

6.2.2 Squashed

For the *squashed* case, identifying along ξ_a we find no horizons. When $\alpha_0^2 < 3(1 - \nu^2)\beta_2^2/4\nu^2$ we find no CTCs either. Identifying along ξ_b always gives rise to CTCs

outside the horizon. Identifying along ξ_c always gives rise to CTCs outside the horizon unless α_0 vanishes in which case we have no CTCs and we have a self-dual type solution.

6.3 Null Warped

The one-parameter subgroups are given by

$$n_a = \alpha_0 N_0 + N \quad (6.10)$$

$$n_b = \alpha_1 N_1 + N \quad (6.11)$$

$$n_c = \alpha_1(N_1 + N_0) + N \quad (6.12)$$

The norms of the Killing vectors generating the subgroups are given in A.29, A.30 and A.31.

6.3.1 Minus Sign

For null warped AdS₃ with a *minus* sign we encounter CTCs at the boundary unless $\alpha_0 = 0$ in which case we have a singularity-free solution.

6.3.2 Plus Sign

For null warped AdS₃ with a *plus* sign, identifying along n_b gives no causal singularity when α_a is positive and gives CTCs outside the horizon when α_1 is negative.

Identifying along n_a we encounter a horizon and causal singularity located at

$$r_h = 0, \quad r_s = \frac{-1 \pm \sqrt{1 - \alpha_0^2}}{2} \quad (6.13)$$

The horizon, r_h , lies outside the causal singularity, r_s , whenever $0 < \alpha_0 < 1$. One can obtain a real coordinate transformation taking the quotiented null warped metric to the following metric

$$\frac{ds^2}{\ell^2} = -\frac{r^2}{r^2 + r + \tilde{\alpha}_0^2} dt^2 + (r^2 + r + \tilde{\alpha}_0^2) \left(d\phi - \frac{rdt}{r^2 + r + \tilde{\alpha}_0^2} \right)^2 + \frac{dr^2}{4r^2} \quad (6.14)$$

where $r \in [r_s, \infty]$, $u \in [-\infty, \infty]$ and $\phi \sim \phi + 2\pi$ and we have defined $\tilde{\alpha}_0 \equiv \alpha_0/2$. The coordinate transformation is given by

$$x^- = e^{\alpha_0 \phi} \quad (6.15)$$

$$x^+ = \phi - 2t - \frac{\alpha_0}{2r} \quad (6.16)$$

$$u = \sqrt{\frac{\alpha_0 e^{\alpha_0 \phi}}{r}} \quad (6.17)$$

We can recognize the above solution as the extremal BC black hole C.1 for the special value $\Omega = \frac{1}{2}$ and $\rho_0 = 3(\nu^2 - 1)\alpha_0^2/4(\nu^2 + 3)$ in the limit $\nu \rightarrow 1$. The conserved ADT charges [15] of this black hole with respect to the ∂_t and ∂_ϕ Killing vectors are given by

$$\mathcal{M}^{ADT} = 0, \quad \mathcal{J}^{ADT} = -\frac{\alpha_0^2 \ell}{12G} \quad (6.18)$$

The temperature and angular momentum at the horizon as defined in 7.12 and 7.13 vanish for the above solution. However the right moving temperature is non-zero:

$$T_R = \frac{\alpha_0}{2\pi\ell} \quad (6.19)$$

and the thermodynamic entropy is given by

$$S = \frac{\pi\alpha_0\ell}{3G} = \frac{\pi^2\ell T_R}{3} \frac{(5\nu^2 + 3)\ell}{G\nu(\nu^2 + 3)} \Big|_{\nu \rightarrow 1} \quad (6.20)$$

The term $(5\nu^2 + 3)\ell/G\nu(\nu^2 + 3)$ will be recognized as the right moving central charge in the following section. A similar situation to n_a holds if we identify along n_c .

7 Thermodynamics

The thermodynamics of spacelike stretched black holes was studied in [15] where it was shown that, after accounting for the effects of the Chern-Simons term on the various thermodynamic quantities, they obey the first law. In this section we reorganize the formulae in a suggestive manner to motivate the conjecture of the concluding section.

7.1 Entropy

The entropy of the warped black hole is composed of two terms. There is the usual term stemming from the Einstein action and a term coming from the Chern-Simons part of the action [47, 48, 49]. The total entropy of the warped black hole is given by

$$S = \frac{\pi\ell}{24\nu G} \left[(9\nu^2 + 3)r_+ - (\nu^2 + 3)r_- - 4\nu\sqrt{(\nu^2 + 3)r_+r_-} \right] \quad (7.1)$$

It is instructive to rewrite this in terms of the temperatures T_L and T_R appearing in 5.9. Defining right and left moving “central charges”,

$$c_R \equiv \frac{(5\nu^2 + 3)\ell}{G\nu(\nu^2 + 3)} \quad (7.2)$$

$$c_L \equiv \frac{4\nu\ell}{G(\nu^2 + 3)} \quad (7.3)$$

allows us to express the entropy of the warped black hole in the following suggestive manner

$$S = \frac{\pi^2\ell}{3} (c_L T_L + c_R T_R). \quad (7.4)$$

This is precisely the formula for the entropy of a 2d CFT with central charges c_L and c_R at temperatures T_L and T_R . Of course the central charges were defined so that this relation holds: a nontrivial observation is that so defined they depend only on the coupling constant ν and not r_\pm . A further nontrivial fact is that the left and right moving central charges, c_L and c_R , are equal to the left and right moving central charges of TMG in AdS₃ when $\nu = 1$. A final nontrivial check is that the difference between the left and right moving central charges matches the coefficient of the diffeomorphism anomaly [50]

$$c_L - c_R = -\frac{\ell}{G\nu} \quad (7.5)$$

One can also define the following left and right moving energies,

$$E_L \equiv \frac{\pi^2\ell}{6} c_L T_L^2 \quad (7.6)$$

$$E_R \equiv \frac{\pi^2\ell}{6} c_R T_R^2 \quad (7.7)$$

Which by construction obey

$$\frac{dS}{dE_L} = \frac{1}{T_L} \quad (7.8)$$

$$\frac{dS}{dE_R} = \frac{1}{T_R}. \quad (7.9)$$

7.2 First law

A black hole is characterized by conserved charges such as the energy or angular momentum, as well as conjugate variables such as temperature or angular potential. It is a nontrivial test of black hole dynamics that these quantities are related by the first law. The spacelike stretched black holes were shown to pass this test in [15]. They compute the so-called ADT mass \mathcal{M}^{ADT} and angular momentum \mathcal{J}^{ADT} , using the surface integral expressions derived in [40, 41, 42, 43] for the conserved charges associated to the asymptotic Killing vectors ∂_t and ∂_θ of 4.1 and find

$$\mathcal{M}^{ADT} = \frac{(\nu^2 + 3)}{24G} \left(r_+ + r_- - \frac{1}{\nu} \sqrt{r_+ r_- (\nu^2 + 3)} \right) \quad (7.10)$$

$$\begin{aligned} \mathcal{J}^{ADT} = \frac{\nu \ell (\nu^2 + 3)}{96G} & \left[\left(r_+ + r_- - \frac{1}{\nu} \sqrt{r_+ r_- (\nu^2 + 3)} \right)^2 \right. \\ & \left. - \frac{(5\nu^2 + 3)}{4\nu^2} (r_+ - r_-)^2 \right] \end{aligned} \quad (7.11)$$

They further compute the Hawking temperature T_H , defined as the surface gravity of the horizon divided by 2π , and the angular velocity of the horizon Ω_H as

$$T_H \equiv \frac{1}{2\pi\ell} \sqrt{g^{rr}} \partial_r N = \frac{(\nu^2 + 3)}{4\pi\ell} \frac{(r_+ - r_-)}{(2\nu r_+ - \sqrt{(\nu^2 + 3)r_+ r_-})} \quad (7.12)$$

$$\Omega_H \equiv \frac{N^\theta}{\ell} = \frac{2}{(2\nu r_+ - \sqrt{(\nu^2 + 3)r_+ r_-})\ell} \quad (7.13)$$

They then explicitly check that these are related to the entropy via the first law:

$$T_H = \left(\frac{\partial S}{\partial \mathcal{M}^{ADT}} \right)^{-1}, \quad \Omega_H = T_H \left(\frac{\partial S}{\partial \mathcal{J}^{ADT}} \right). \quad (7.14)$$

In the preceding subsection we propose that the conserved charges (E_L, E_R) and potentials (T_L, T_R) are more natural for this system than $(\mathcal{M}^{ADT}, \mathcal{J}^{ADT})$ and

T_H, Ω_H . Since this is just a change of variables - albeit an illuminating one - the first law will still hold for the new charges/potentials. To see this explicitly one can express the conserved ADT charges in terms of the left and right moving temperatures and energies as follows,

$$\mathcal{M}^{ADT} = \frac{1}{G} \sqrt{\frac{2\ell E_L}{3c_L}} \quad (7.15)$$

$$\mathcal{J}^{ADT} = \ell(E_L - E_R). \quad (7.16)$$

The potentials are related by

$$\frac{1}{T_H} = \frac{4\pi\nu\ell}{\nu^2 + 3} \frac{T_L + T_R}{T_R} \quad (7.17)$$

$$\frac{\Omega_H}{T_H} = \frac{1}{T_R\ell}. \quad (7.18)$$

One may then readily use these relations to verify that 7.14 is equivalent to the first law 7.9 in the new variables.⁷

7.3 A conjecture

Topologically massive gravity with $\nu > 1$ ($\mu\ell > 3$) admits an $U(1)_L \times SL(2, \mathbb{R})_R$ -invariant candidate ground state referred to as spacelike stretched (warped) AdS₃, as well as regular black hole solutions. We conjecture that with suitable asymptotically stretched AdS₃ boundary conditions $\nu > 1$ quantum TMG is holographically dual to a 2D boundary CFT with $c_R = \frac{(5\nu^2+3)\ell}{G\nu(\nu^2+3)}$ and $c_L = \frac{4\nu\ell}{G(\nu^2+3)}$.

The primary motivations for this conjecture are that application of the Cardy formula to the CFT density of states reproduces the black hole entropy, and that as far as we understand the conjecture does not contradict any known properties of the theory. The conjecture passed some weak tests in section 7. Further tests of the conjecture are possible. After understanding the boundary conditions, perturbative stability must be demonstrated. These boundary conditions will also determine the asymptotic symmetry group, and perhaps enable a check of the formulae 7.2 and 7.3 for the central charges [51] along the lines of [37]. Finally, it might be possible to engineer TMG on warped AdS₃ in a decoupling limit of string theory and find the dual CFT.

⁷We note that similar formulae work for the null black holes which arise in the $\nu \rightarrow 1$ limit and have $T_L=0$.

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A Isometries

We write down the relevant Killing vectors for the various warped AdS_3 metrics. The $SL(2, \mathbb{R})_L$ isometries are given by

$$J_1 = \frac{2 \sinh u}{\cosh \sigma} \partial_\tau - 2 \cosh u \partial_\sigma + 2 \tanh \sigma \sinh u \partial_u \quad (\text{A.1})$$

$$J_2 = 2 \partial_u \quad (\text{A.2})$$

$$J_0 = -\frac{2 \cosh u}{\cosh \sigma} \partial_\tau + 2 \sinh u \partial_\sigma - 2 \tanh \sigma \cosh u \partial_u \quad (\text{A.3})$$

These satisfy the algebra $[J_1, J_2] = 2J_0$, $[J_0, J_1] = -2J_2$ and $[J_0, J_2] = 2J_1$. The $SL(2, \mathbb{R})_R$ isometries are given by

$$\tilde{J}_1 = 2 \sin \tau \tanh \sigma \partial_\tau - 2 \cos \tau \partial_\sigma + \frac{2 \sin \tau}{\cosh \sigma} \partial_u \quad (\text{A.4})$$

$$\tilde{J}_2 = -2 \cos \tau \tanh \sigma \partial_\tau - 2 \sin \tau \partial_\sigma - \frac{2 \cos \tau}{\cosh \sigma} \partial_u \quad (\text{A.5})$$

$$\tilde{J}_0 = 2 \partial_\tau \quad (\text{A.6})$$

These satisfy the algebra $[\tilde{J}_1, \tilde{J}_2] = 2\tilde{J}_0$, $[\tilde{J}_0, \tilde{J}_1] = -2\tilde{J}_2$ and $[\tilde{J}_0, \tilde{J}_2] = 2\tilde{J}_1$.

As mentioned in the text the Killing vectors preserved by spacelike warped anti-de Sitter space are given by the $SL(2, \mathbb{R})_R$ and J_2 . The Killing vectors preserved by timelike warped anti-de Sitter space are given by the $SL(2, \mathbb{R})_L$ and \tilde{J}_0 .

The Killing vectors for null warped anti-de Sitter space are given by

$$N_1 = \partial_-, \quad (\text{A.7})$$

$$N_0 = x^- \partial_- + \frac{u}{2} \partial_u, \quad (\text{A.8})$$

$$N_{-1} = (x^-)^2 \partial_- - u^2 \partial_+ + x^- u \partial_u \quad (\text{A.9})$$

$$N = \partial_+ \quad (\text{A.10})$$

where the $U(1)_{null}$ is spanned by N . These satisfy the algebra $[N_1, N_0] = N_1$, $[N_{-1}, N_0] = -N_{-1}$ and $[N_1, N_{-1}] = 2N_0$.

A.1 Classification of $SL(2, \mathbb{R}) \times U(1)$ Subgroups

In this section we classify the various one-parameter subgroups of the isometries for the three types of warped anti-de Sitter space. Our classification is based on the classification of the one-parameter subgroups of $SL(2, \mathbb{R})_L \times SL(2, \mathbb{R})_R$ given in [17].

A.1.1 Spacelike Warped anti-de Sitter Space

Spacelike warped AdS has isometry group $U(1)_L \times SL(2, \mathbb{R})_R$. The most general form of the Killing vector along which we can quotient is

$$\beta_2 J_2 + \alpha_2 \tilde{J}_2 + \alpha_0 \tilde{J}_0 + \alpha_1 \tilde{J}_1. \quad (\text{A.11})$$

We can always get rid of the α_1 via a $U(1)_L \times SL(2, \mathbb{R})_R$ transformation. There are three types of one parameter subgroups generated by

$$\eta_a : \quad \beta_2 J_2 + \alpha_0 \tilde{J}_0 \quad (\text{A.12})$$

$$\eta_b : \quad \beta_2 J_2 + \alpha_2 \tilde{J}_2 \quad (\text{A.13})$$

$$\eta_c : \quad \beta_2 J_2 + \tilde{J}_0 + \tilde{J}_2 \quad (\text{A.14})$$

In terms of the classification in [17], these correspond to I_a , I_b , and II_a (the first form) respectively. The norms of the above generators are

$$\begin{aligned} \eta_a^2 &= \frac{12\ell^2(\nu^2 - 1)}{(\nu^2 + 3)^2} [\alpha_0 \sinh(\sigma)]^2 + \frac{32\nu^2\ell^2}{(\nu^2 + 3)^2} \beta_2 [\alpha_0 \sinh(\sigma)] \\ &\quad - \frac{4\ell^2}{\nu^2 + 3} \alpha_0^2 + \frac{16\nu^2\ell^2}{(\nu^2 + 3)^2} \beta_2^2 \end{aligned} \quad (\text{A.15})$$

$$\begin{aligned} \eta_b^2 &= \frac{12\ell^2(\nu^2 - 1)}{(\nu^2 + 3)^2} [\alpha_2 \cos(\tau) \cosh(\sigma)]^2 - \frac{32\nu^2\ell^2}{(\nu^2 + 3)^2} \beta_2 [\alpha_2 \cos(\tau) \cosh(\sigma)] \\ &\quad + \frac{4\ell^2}{\nu^2 + 3} (\alpha_2^2) + \frac{16\nu^2\ell^2}{(\nu^2 + 3)^2} \beta_2^2 \end{aligned} \quad (\text{A.16})$$

$$\begin{aligned} \eta_c^2 &= \frac{12\ell^2(\nu^2 - 1)}{(\nu^2 + 3)^2} [\sinh(\sigma) - \cos(\tau) \cosh(\sigma)]^2 + \frac{32\nu^2\ell^2}{(\nu^2 + 3)^2} \beta_2 [\sinh(\sigma) - \cos(\tau) \cosh(\sigma)] \\ &\quad + \frac{16\nu^2\ell^2}{(\nu^2 + 3)^2} \beta_2^2 \end{aligned} \quad (\text{A.17})$$

A.1.2 Timelike Warped anti-de Sitter Space

For timelike warped, the most general form of the Killing vector is

$$\alpha_0 \tilde{J}_0 + \beta_2 J_2 + \beta_0 J_0 + \beta_1 J_1. \quad (\text{A.18})$$

We can always eliminate β_1 via an $SL(2, \mathbb{R})_L \times U(1)_R$ transformation. There are three types of one parameter subgroups generated by

$$\xi_a : \quad \alpha_0 \tilde{J}_0 + \beta_2 J_2 \quad (\text{A.19})$$

$$\xi_b : \quad \alpha_0 \tilde{J}_0 + \beta_0 J_0 \quad (\text{A.20})$$

$$\xi_c : \quad \alpha_0 \tilde{J}_0 + J_0 + J_2 \quad (\text{A.21})$$

In terms of the classification in [17], these correspond to I_a , I_b , and II_a (the first form) respectively. The norms are

$$\begin{aligned} \xi_a^2 &= -\frac{12\ell^2(\nu^2-1)}{(\nu^2+3)^2} [\beta_2 \sinh(\sigma)]^2 - \frac{32\nu^2\ell^2}{(\nu^2+3)^2} \alpha_0 [\beta_2 \sinh(\sigma)] \\ &\quad + \frac{4\ell^2}{\nu^2+3} (\beta_2^2) - \frac{16\nu^2\ell^2}{(\nu^2+3)^2} \alpha_0^2 \end{aligned} \quad (\text{A.22})$$

$$\begin{aligned} \xi_b^2 &= -\frac{12\ell^2(\nu^2-1)}{(\nu^2+3)^2} [\beta_0 \cosh(u) \cosh(\sigma)]^2 + \frac{32\nu^2\ell^2}{(\nu^2+3)^2} \alpha_0 [\beta_0 \cosh(u) \cosh(\sigma)] \\ &\quad - \frac{4\ell^2}{\nu^2+3} (\beta_0^2) - \frac{16\nu^2\ell^2}{(\nu^2+3)^2} \alpha_0^2 \end{aligned} \quad (\text{A.23})$$

$$\begin{aligned} \xi_c^2 &= -\frac{12\ell^2(\nu^2-1)}{(\nu^2+3)^2} [\sinh(\sigma) - \cosh(u) \cosh(\sigma)]^2 - \frac{32\nu^2\ell^2}{(\nu^2+3)^2} \alpha_0 [\sinh(\sigma) - \cosh(u) \cosh(\sigma)] \\ &\quad - \frac{16\nu^2\ell^2}{(\nu^2+3)^2} \alpha_0^2 \end{aligned} \quad (\text{A.24})$$

A.1.3 Null Warped anti-de Sitter Space

The most general linear combination of Killing vectors is

$$\xi = N + \alpha_1 N_1 + \alpha_0 N_0 + \alpha_{-1} N_{-1} \quad (\text{A.25})$$

By a shift of x^- , we can eliminate the term with N_{-1} . Then the distinct subgroups are generated by

$$n_a = \alpha_0 N_0 + N \quad (\text{A.26})$$

$$n_b = \alpha_1 N_1 + N \quad (\text{A.27})$$

$$n_c = \alpha_1 (N_1 + N_0) + N \quad (\text{A.28})$$

We have not included a coefficient in front of the N because it is null. The norm is

$$n_a^2 = \ell^2 \left(\pm \frac{\alpha_0^2(x^-)^2}{u^4} + \frac{\alpha_0}{u^2}(x^-) + \frac{\alpha_0^2}{4} \right) \quad (\text{A.29})$$

$$n_b^2 = \ell^2 \left(\pm \frac{\alpha_1^2}{u^4} + \frac{\alpha_1}{u^2} \right) \quad (\text{A.30})$$

$$n_c^2 = \ell^2 \left(\pm \frac{\alpha_1^2}{u^4}(1+x^-)^2 + \frac{\alpha_1}{u^2}(1+x^-) + \frac{\alpha_1^2}{4} \right) \quad (\text{A.31})$$

where the plus sign corresponds to the null warped metric with a plus sign and vice versa.

B Extremal Black Holes

It is convenient to introduce spacelike warped AdS₃ in Poincare coordinates

$$\frac{ds^2}{\ell^2} = \frac{1}{\nu^2 + 3} \left(-x^2 d\psi^2 + \frac{dx^2}{x^2} + \frac{4\nu^2}{\nu^2 + 3} (d\phi + x d\psi)^2 \right) \quad (\text{B.1})$$

and note that the coordinate transformation from Poincare coordinates to the global coordinate in 3.3 is:

$$\psi = -\frac{\sin(\tau)}{2(\tanh(\sigma) - \cos(\tau))} \quad (\text{B.2})$$

$$\phi = u + 2 \tanh^{-1} \left(e^\sigma \tan \left(\frac{\tau}{2} \right) \right) \quad (\text{B.3})$$

$$x = 2(\sinh(\sigma) - \cos(\tau) \cosh(\sigma)) \quad (\text{B.4})$$

The metric of extremal warped black hole is obtained by setting $r_+ = r_- = r_h$ in 4.1 to obtain

$$\frac{ds^2}{\ell^2} = dt^2 + \frac{dr^2}{(\nu^2 + 3)(r - r_h)^2} + \left(2\nu r - \sqrt{\nu^2 + 3} r_h \right) dt d\theta \quad (\text{B.5})$$

$$+ \frac{r}{4} \left(3(\nu^2 - 1)r + 2(\nu^2 + 3)r_h - 4\nu r_h \sqrt{\nu^2 + 3} \right) d\theta^2 \quad (\text{B.6})$$

Then the coordinate transformation from the Poincare coordinate to the extremal black hole is

$$t = \frac{2\nu}{(\nu^2 + 3)}\phi + \frac{r_h(-2\nu + \sqrt{\nu^2 + 3})\psi}{(\nu^2 + 3)} \quad (\text{B.7})$$

$$\theta = \frac{2\psi}{\nu^2 + 3} \quad (\text{B.8})$$

$$r = x + r_h \quad (\text{B.9})$$

Combining the above coordinate transformations gives the coordinate transformation between warped AdS₃ and the extremal warped black hole,

$$t = \frac{2\nu}{(\nu^2 + 3)} \left(u + 2 \tanh^{-1} \left(e^\sigma \tan \left(\frac{\tau}{2} \right) \right) \right) - \frac{r_h(-2\nu + \sqrt{\nu^2 + 3}) \sin(\tau)}{2(\nu^2 + 3)(\tanh(\sigma) - \cos(\tau))} \quad (\text{B.10})$$

$$\theta = -\frac{\sin(\tau)}{(\nu^2 + 3)(\tanh(\sigma) - \cos(\tau))} \quad (\text{B.11})$$

$$r = 2(\sinh(\sigma) - \cos(\tau) \cosh(\sigma)) + r_h \quad (\text{B.12})$$

C BC Coordinates

C.1 Metric

In [15] a warped black hole closely related to the Schwarzschild solution in 4.1 was obtained by a simple transformation which we describe below. We will refer to these coordinates as BC coordinates for which the metric is given by

$$\frac{ds_{BC}^2}{\ell^2} = -\frac{(\nu^2 + 3)}{4\nu^2} \frac{\rho^2 - \rho_0^2}{R(\rho)^2} dT^2 + \frac{1}{(\nu^2 + 3)} \frac{d\rho^2}{\rho^2 - \rho_0^2} + R(\rho)^2 \left[d\phi - \frac{4\nu^2 \rho + 3(\nu^2 - 1)\Omega}{4\nu^2 R(\rho)^2} dT \right]^2 \quad (\text{C.1})$$

where $\rho \in [\rho_s, \infty]$, $T \in [-\infty, \infty]$ and $\phi \sim \phi + 2\pi$. We have defined

$$R(\rho)^2 \equiv \rho^2 + 2\Omega\rho + \frac{3\Omega^2(\nu^2 - 1)}{4\nu^2} + \frac{(\nu^2 + 3)}{3(\nu^2 - 1)}\rho_0^2 \quad (\text{C.2})$$

such that $R(\rho_s) = 0$. There is a vacuum solution analogous to the massless BTZ or Poincare anti-de Sitter space for the BC black hole which is given by setting $\rho_0 = \Omega = 0$. The horizons of the BC black hole are located at $\rho = \pm\rho_0$ where $\rho_0 > 0$ and the (causal) singularities are located at $\rho = \rho_s$. We would like to emphasize that these are singularities in the causal structure [17] and not spacetime singularities that appear in the coordinate invariants. The metric C.1 contains such causal singularities

in the parameter range

$$\Omega^2 > \frac{4\nu^2\rho_0^2}{3(\nu^2 - 1)} \quad (\text{C.3})$$

In order to avoid naked singularities we require that $\rho_0 > \rho_s$ which gives us the bound

$$\frac{3(\nu^2 - 1)}{4\nu^2} \left[\Omega + \frac{4\nu^2\rho_0}{3(\nu^2 - 1)} \right]^2 > 0 \quad (\text{C.4})$$

This condition is satisfied for $\nu^2 > 1$ and thus the physical black holes for both the Schwarzschild and BC coordinates are in the same parameter region. The location of the horizon and singularity coincide whenever $\Omega = -4\nu^2\rho_0/3(\nu^2 - 1)$.

C.2 Schwarzschild to BC Coordinate Transformation

The coordinate transformation relating the solution in Schwarzschild coordinates 4.1 to the BC solution C.1 is given by

$$t = \frac{1}{\nu} \left(\frac{3(\nu^2 - 1)}{4} \right)^{1/2} T \quad (\text{C.5})$$

$$\theta = - \left(\frac{4}{3(\nu^2 - 1)} \right)^{1/2} \phi \quad (\text{C.6})$$

$$r = \rho + \frac{1}{2}(r_+ + r_-) \quad (\text{C.7})$$

We note that the transformation is singular when $\nu^2 = 1$. When $\nu^2 < 1$ the transformation becomes imaginary. This corresponds to the parameter space containing unphysical black holes. The relation between the parameters of the Schwarzschild solution and the parameters of the BC solution can also be obtained quite simply and is given by

$$\rho_0^2 = \frac{1}{4}(r_+ - r_-)^2 \quad (\text{C.8})$$

$$\Omega = \frac{2\nu^2}{3(\nu^2 - 1)} \left(r_+ + r_- - \frac{\sqrt{r_+ r_-(\nu^2 + 3)}}{\nu} \right) \quad (\text{C.9})$$

such that $\rho_s = -(r_+ + r_-)/2$.

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